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LINEAR DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

BY MAXIME BÔCHER

IF we consider the problem of determining the simple harmonic vibrations of a stretched wire whose cross section changes abruptly at some points, or the analogous problem of the flow of heat in a rod of similar character, we are led to a linear differential equation of the second order whose coefficients are discontinuous functions of the real variable x. Such "finite discontinuities," or even much more complicated ones,* in the coefficients of the differential equation should not be regarded as being in any true sense singular points of the equation, the only peculiarity which they produce in the solutions being a discontinuity in the second derivative. The same is true when the coefficients of the differential equation have discontinuities where they fail to remain finite, provided merely that the integrals of the absolute values of the coefficients converge when extended up to such points of discontinuity. The proofs of the fundamental theorems concerning linear differential equations can be given in such a form as to cover all these cases, and, a fact to which I wish especially to call attention, this greater generality does not involve any material increase in complexity or difficulty of the proof. In the following pages I have given such a treatment, confining myself for the sake of simplicity to equations of the second order though the method used admits of immediate and obvious extension to linear differential equations of any order.

1. General preliminaries. We shall be concerned exclusively with real values of x which lie in the interval

$$(I) \qquad a \leqq x \leqq b.$$

While some of the functions with which we shall deal are continuous throughout (I), others are of a more general character. These we designate as *integrable* according to the

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* As for instance the discontinuity which $\sin 1/x$ has at the point $x = 0$

DEFINITION. *A function $f(x)$ is said to be integrable throughout (I) if it is continuous except at a finite number of points, and if $\int f(x)dx$ converges when extended over any portion of (I).**

The following facts concerning integrable functions are well known, or can be easily proved:

1) If $\phi(x)$ has only a finite number of discontinuities and $|\phi(x)|$ is integrable, then $\phi(x)$ is integrable. If this is the case we shall speak of ϕ as *absolutely integrable.*

2) If $\phi_1, \phi_2, \cdots \phi_k$ are integrable (or absolutely integrable), the function $\phi_1 + \phi_2 + \cdots + \phi_k$ is integrable (or absolutely integrable).

3) If $\phi(x)$ is integrable, $\int_c^x \phi(x)dx$ is continuous.

4) If $\phi(x)$ is absolutely integrable and $f(x)$ is continuous, $f\cdot\phi$ is absolutely integrable.

5) If $\phi(x)$ is integrable and $f(x)$ is continuous and has a continuous (or absolutely integrable) derivative, $f\cdot\phi$ is integrable. (Proof by integration by parts.)

It is in no way necessary, except in §5, to require that the functions with which we deal be real, although, as has been said, the independent variable x is supposed real. By slight and obvious changes the whole treatment can be carried over to the case of analytic functions of complex variables. This has been indicated briefly in a simple case in §4, where *one* of the independent variables (namely λ) is allowed to become complex. Only slight changes would be necessary in order to allow x also to become complex.

2. The Existence Theorem.†

We will consider the differential equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x), \tag{1}$$

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*It should be carefully noticed that this definition is more general than Riemann's in that it does not require the function to remain finite in (I), but less general in that it admits only a finite number of points of discontinuity.

†*Cf.* the earlier papers of the writer: *Bulletin of the American Mathematical Society*, March, 1899; *American Journal of Mathematics*, vol. 24, 1902, where this same question is treated by slightly different methods.

and assume that r is an integrable function throughout (I) and p and q are absolutely integrable there.*

The question now arises what we should understand by a solution of (1), the difficulty being that we can clearly not demand that a function which satisfies (1) at other points should also satisfy it when one or more of the functions p, q, r become discontinuous. If we make no demand whatever on the solution at such a point of discontinuity of the coefficients of (1), there will clearly be no connection between the values of the solution on the two sides of this point. It is desirable for most purposes to demand that the solution itself and its first derivative be continuous at such points. Thus we lay down the following

DEFINITION. *By a solution of* (1) *is understood a function of x which at every point of (I) is continuous and has a continuous first derivative, and at every point of (I) where p, q, r are continuous has a second derivative and satisfies* (1).

We now establish the existence theorem:

If c is any point of (I) and γ, γ_1 any constants, there exists a solution of (1) *which satisfies the auxiliary conditions*

$$y(c) = \gamma, \quad y'(c) = \gamma_1. \tag{2}$$

Let us establish this theorem first in the special case in which p and q vanish at every point of (I). The general solution of (1) is then

$$y = \int_c^x \left(\int_c^x r(x)\,dx \right) dx + A(x-c) + B,$$

a function which is obviously continuous and has a continuous first derivative throughout (I). If we let $A = \gamma_1$, $B = \gamma$, conditions (2) are satisfied. Thus the theorem is proved in this special case. The expression for y which we have obtained can be thrown into a more convenient form by an integration by parts, namely:

$$y = x\int_c^x r(x)\,dx - \int_c^x xr(x)\,dx + \gamma_1(x-c) + \gamma.$$

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* It is not unlikely that it would be sufficient to require that p and q be merely integrable The writer would be glad to learn of a method by which this more general case could be treated.

By a slight change of notation we get the following expression alongside of which we write for reference the value of y':

$$(3)\qquad \begin{cases} y = \displaystyle\int_c^x (x-\xi)r(\xi)d\xi + \gamma_1(x-c) + \gamma, \\ y' = \displaystyle\int_c^x r(\xi)\,d\xi + \gamma_1. \end{cases}$$

We now treat the general case by the method of successive approximations. Starting from the value $y_0 = 0$, we compute the successive approximations $y_1, y_2, \cdots$ by means of the recurrent relation:

$$y''_n = -py'_{n-1}, - qy_{n-1} + r \qquad (n = 1, 2, \cdots),$$

and the auxiliary conditions

$$y_n(c) = \gamma \quad y'_n(c) = \gamma_1.$$

We shall prove that y_n approaches a limit as n becomes infinite, and that this limit is the solution of (1) whose existence we wish to establish.

If we write

$$u_n(x) = y_{n+1}(x) - y_n(x),$$

y_n is obviously the sum of the first n terms of the series

$$(4)\qquad u_0(x) + u_1(x) + u_2(x) + \cdots.$$

We must then prove that this series converges and represents a solution of (1) which satisfies conditions (2). It is convenient to consider by the side of (4) the series

$$(5)\qquad u'_0(x) + u'_1(x) + u'_2(x) + \cdots.$$

Since u_0 and y_1 are identical, we see that u_0 and u'_0 are given directly by formulæ (3). The subsequent u's may be determined by the recurrent relation

$$u''_n = -p(x)\; u'_{n-1} - q(x)u_{n-1} \qquad (n = 1, 2, \cdots)$$

with the auxiliary conditions

$$u_n(c) = 0, \quad u'_n(c) = 0,$$

as we see by subtracting from one another the recurrent relations and also the auxiliary conditions for determining y_{n+1} and y_n. Applying (3) we now find

$$(6) \qquad \begin{cases} u_n(x) = -\int_c^x (x-\xi)\Big(p(\xi)u'_{n-1}(\xi) + q(\xi)u_{n-1}(\xi)\Big)d\xi, \\ u'_n(x) = -\int_c^x \Big(p(\xi)u'_{n-1}(\xi) + q(\xi)u_{n-1}(\xi)\Big)d\xi. \end{cases}$$

Thus we see that all the terms of (4) and (5) are continuous functions of x throughout (I).

Let us now introduce a positive constant l subject to the conditions

$$(7) \qquad l \geqq 1, \qquad l \geqq b - a.$$

We then infer from (6) the inequality

$$(8) \qquad \left.\begin{matrix} |u_n(x)| \\ |u'_n(x)| \end{matrix}\right\} \leqq l\int_c^x \Big(|p(\xi)|+|q(\xi)|\Big)\Big(|u'_{n-1}(\xi)|+|u_{n-1}(\xi)|\Big)|d\xi| \qquad (n = 1, 2, \cdots).$$

Since $u_0(x)$ and $u'_0(x)$ are continuous throughout (I), there exists a positive constant C such that

$$(9) \qquad |u_0(x)| \leqq C, \qquad |u'_0(x)| \leqq C.$$

We are now in a position to prove the fundamental inequalities

$$(10) \qquad \left.\begin{matrix} |u_n(x)| \\ |u'_n(x)| \end{matrix}\right\} \leqq C\frac{\left[2l\int_c^x \Big(|p(\xi)|+|q(\xi)|\Big)|d\xi|\right]^n}{n!}.$$

Since (10) reduces to (9) when $n = 0$, it is sufficient to prove that if it is true when $n = k$, it is true when $n = k + 1$.

Letting $n = k + 1$ in (8), and replacing $|u_k(\xi)|$ and $|u_k'(\xi)|$ in the second member of (8) by values obtained by letting $n = k$ in (10), we get

$$\left.\begin{matrix} |u_{k+1}(x)| \\ |u'_{k+1}(x)| \end{matrix}\right\} \leqq C\,\frac{(2l)^{k+1}}{k!}\int_c^x \Big(|p(\xi)|+|q(\xi)|\Big)\left[\int_c^\xi \Big(|p(\xi)|+|q(\xi)|\Big)|d\xi|\right]^k |d\xi|.$$

From this we deduce the special case of (10) which we wish to establish by making use of the formula

$$\int_c^x \phi(\xi)\Big(\int_c^\xi \phi(\xi)d\xi\Big)^k d\,\xi = \frac{1}{k+1}\Big(\int_c^x \phi(\xi)d\xi\Big)^{k+1},$$

whose correctness follows by an integration by parts.

If now we let

$$M=\int_a^b\Big(|p(\xi)|+|q(\xi)|\Big)d\xi,$$

we obtain from (10) the simpler inequalities

$$\left.\begin{array}{l}|u_n(x)|\\ |u_n'(x)|\end{array}\right\}\leqq C\,\frac{(2lM)^n}{n\,!}. \tag{11}$$

These inequalities show that at every point of (I) the terms of the serie (4) and (5) do not exceed in absolute value the terms of the series

$$C+C\,\frac{2lM}{1\,!}+C\,\frac{(2lM)^2}{2\,!}+\cdots,$$

and, since this is a convergent series of positive constant terms, it follows by Weierstrass's fundamental test that the series (4) and (5) are uniformly convergent throughout (I), and therefore, since their terms are continuous, represent continuous functions throughout this interval. Moreover, by a well known theorem concerning uniform convergence, we see that (5) represents the derivative of (4).

Denoting by $y(x)$ the function represented by (4), it is clear that y satisfies conditions (2). In order to prove that it is the solution of (1) whose existence we wish to establish it is then merely necessary to prove that y satisfies (1) at every point of (I) where p, q, r are continuous.

Let x_0 be any such point, and surround it by an interval (J) lying within (I) throughout which p, q, r are continuous. If we multiply (5) by $-p(x)$, (4) by $-q(x)$ and add the two resulting series together, we get, as we see by referring to the differential equation for u_n,

$$-p(x)\,y'(x)-q(x)\,y(x)=u_1''(x)+u_2''(x)+\cdots \tag{12}$$

From the way in which this series was obtained it is clear that it is uniformly convergent throughout (J) and that its terms are continuous there. Accordingly, since it may also be obtained by differentiating (5) term by term (the first term being omitted) its value is

$$y''(x)-u_0''(x)=y''(x)-r(x).$$

Substituting this value in (12), we see that y satisfies (1) throughout (J).

That the solution whose existence we have established is unique is stated by the following theorem which we now proceed to prove:

If two solutions of (1) *satisfy conditions* (2), *they are identically equal throughout* (I).

The difference of two such solutions is a solution of the homogeneous equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0, \tag{13}$$

satisfying the auxiliary conditions

$$y(c) = 0, \qquad y'(c) = 0. \tag{14}$$

Hence our theorem will be established if we can prove the following more special result:

A solution y_1 *of* (13), *which satisfies the auxiliary conditions* (14) *vanishes at every point of* (I).

We prove this by a method due to Sturm.* Let x_0 be any point of (I), and consider a solution y_2 of (13) satisfying the auxiliary conditions

$$y_2(x_0) = 0, \quad y'_2(x_0) = 1.$$

By Abel's formula

$$W(x) = y_1(x)y_2'(x) - y_2(x)y'_1(x) = ke^{-\int_c p\,dx},$$

where k is a constant. Since $W(c) = 0$, we see that $k = 0$. Accordingly $W(x_0) = 0$. But $W(x_0) = y_1(x_0)$. Thus we see that y_1 vanishes at x_0 which was any point of (I).

3. Equations which involve parameters.†

It often happens that the coefficients p, q, r of (1) involve besides the variable x one or more parameters, and it is then important to know the nature of the dependence of the solutions of (1) on these parameters. In what follows we confine our attention to the case of a single parameter, but the methods and results can be immediately extended to the case of any number of parameters.

We consider the differential equation

$$\frac{d^2y}{dx^2} + p(x, \lambda)\frac{dy}{dx} + q(x, \lambda)y = r(x, \lambda), \tag{15}$$

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* *Liouville's Journal*, vol. 1 (1836), p. 109.

† Instead of allowing the coefficients of (1) to vary through the introduction of one or more parameters, we may, without any essential complication, adopt the much more general standpoint of the calculus of variations. *Cf.* the paper in the *American Journal* already referred to. The conditions imposed there, at the bottom of p. 315, on the variation of the coefficients might easily be made much less restrictive, as will be seen by comparison with the present paper.

where the real parameter λ is confined to the interval

$$(L) \qquad l_1 \leqq \lambda \leqq l_2.$$

We shall speak of the pair of values (x,λ) as a *point* of the region (I, L) if x is a point of (I) and λ a point of (L). We impose on the coefficients of (15) the following conditions:

(A) *For every value of λ in (L) r is integrable and p and q are absolutely integrable throughout (I). Furthermore the function*

$$\int_c^x r(x,\lambda)dx$$

is a continuous function of the two independent variables (x, λ) at every point of (I, L). Finally, λ_1 being any fixed point and λ_2 a variable point in (L):

$$(16) \qquad \begin{aligned} \lim_{\lambda_2 = \lambda_1} \int_a^b | p(x, \lambda_2) - p(x, \lambda_1) | \, dx = 0, \\ \lim_{\lambda_2 = \lambda_1} \int_a^b | q(x, \lambda_2) - q(x, \lambda_1) | \, dx = 0.* \end{aligned}$$

The fundamental theorem is the following where we append an accent to a function of two variables to denote differentiation with regard to the *first* argument.

If $\gamma(\lambda)$ and $\gamma_1(\lambda)$ are continuous functions of λ throughout (L), and c is any point of (I), and the coefficients of (15) satisfy conditions (A), then the solution $y(x, \lambda)$ of (15) determined by the auxiliary conditions

$$y(c, \lambda) = \gamma(\lambda), \quad y'(c, \lambda) = \gamma_1(\lambda)$$

is a continuous function of the two independent variables (x, λ) throughout (I, L); and the same is true of $y'(x, \lambda)$.

This theorem will be proved if we can show that all the terms of the series (4) and (5) are continuous functions of (x, λ) throughout (I, L), and that these series converge uniformly in this region.

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* It can readily be shown that when conditions (16) are fulfilled the integrals

$$\int_c^x p(x, \lambda)dx, \qquad \int_c^x q(x, \lambda)dx$$

are continuous functions of (x, λ). It is an open question whether (16) might not be replaced by the less restrictive conditions that the two integrals last written be continuous functions of (x, λ). *Cf.* foot-note p. 51.

In establishing the continuity of these terms it will evidently be sufficient to consider the functions u'_i since the functions u_i may be written

$$u_0 = \int_c^x u'_0 \, dx + \gamma(\lambda),$$

$$u_i = \int_c^x u'_i \, dx \qquad (i = 1, 2, \cdots),$$

so that their continuity as functions of (x, λ) follows from the continuity of the u''s.

Since the function u'_0 is given by the second formula (3), it is obviously a continuous function of (x, λ) on account of the restriction placed on r in condition (A). Accordingly we use the method of mathematical induction, assuming that it has been proved that u_{n-1} and u'_{n-1} are continuous functions of (x, λ) and then proving that u'_n is continuous. For this purpose we use the value of u'_n given in the second formula (6) which may be written as the negative of the sum of the two integrals

$$(17) \qquad \int_c^x p(\xi, \lambda) u'_{n-1}(\xi, \lambda) d\xi, \qquad \int_c^x q(\xi, \lambda) u_{n-1}(\xi, \lambda) d\xi.$$

We wish to prove that these two integrals are continuous functions of (x, λ). Denoting the first of them by $F(x, \lambda)$ we must then show that

$$\Delta F = F(x + \Delta x, \lambda + \Delta\lambda) - F(x, \lambda)$$

approaches zero as its limit as Δx and $\Delta\lambda$ approach zero independently of each other. Now we have:

$$\begin{aligned} \Delta F = \int_c^{x+\Delta x} & [p(\xi, \lambda + \Delta\lambda) - p(\xi, \lambda)] u'_{n-1}(\xi, \lambda + \Delta\lambda) d\xi \\ & + \int_c^{x+\Delta x} p(\xi, \lambda) [u'_{n-1}(\xi, \lambda + \Delta\lambda) - u'_{n-1}(\xi, \lambda)] d\xi, \\ & + \int_x^{x+\Delta x} p(\xi, \lambda) u'_{n-1}(\xi, \lambda) d\xi. \end{aligned}$$

We will denote these three integrals by ϕ_1, ϕ_2, ϕ_3 respectively, and we will show that they all approach zero as Δx and $\Delta\lambda$ approach zero.

Since u'_{n-1} is continuous, a positive constant K can be chosen so that throughout (I, L)

$$|u'_{n-1}(x, \lambda)| \leqq K.$$

Accordingly we have

$$|\phi_1| \leqq K \int_c^{x+\Delta x} |p(\xi, \lambda+\Delta\lambda) - p(\xi, \lambda)|\,|d\xi|$$
$$\leqq K \int_a^b | p(\xi, \lambda+\Delta\lambda) - p(\xi, \lambda)|\, d\xi,$$

and this is seen to approach zero by (16). Furthermore,

$$|\phi_2| \leqq \int_a^b |p(\xi, \lambda)|\,|u'_{n-1}(\xi, \lambda+\Delta\lambda) - u'_{n-1}(\xi, \lambda)|\, d\xi \leqq \eta \int_a^b |p(\xi, \lambda)|\, d\xi,$$

where η is the greatest value of $u'_{n-1}(x, \lambda+\Delta\lambda) - u'_{n-1}(x, \lambda)$ in (I). Since η approaches the limit zero as $\Delta\lambda$ approaches zero, ϕ_2 does the same.

Finally in considering ϕ_3 let us notice that since p is absolutely integrable throughout (I) and u'_{n-1} is continuous there, pu'_{n-1} is absolutely integrable there, and hence its integral ϕ_3 is a continuous function of the upper limit of integration, *i. e.* of Δx. This shows that ϕ_3 approaches the limit zero with Δx.

We have thus proved that the first of the integrals (17) is a continuous function of (x, λ). The same proof applies to the second if we interchange p and q, and u'_{n-1} and u_{n-1}. The proof that all the terms of (4) and (5) are continuous functions of (x, λ) is thus complete.

To show that these series converge uniformly in (I, L) we have merely to repeat with very slight changes the reasoning given in §2 for establishing their uniform convergence in (I). Thus the constant C must be so chosen that (9) holds throughout (I, L), and this is possible since u_0 and u'_0 are now continuous functions of (x, λ) throughout this region. Furthermore

$$\int_a^b \Big(|p(\xi, \lambda)| + |q(\xi, \lambda)| \Big)\, d\xi$$

is now a function of λ. It is easily seen by (16) to be continuous throughout (L), and therefore we can introduce a positive constant M so that

$$M \geqq \int_a^b \Big(|p(\xi, \lambda)| + |q(\xi, \lambda)| \Big) d\xi.$$

Formula (11) will then hold and the proof of uniform convergence is complete.

The condition (A) which we have imposed on the coefficients p, q, r is so general that it is worth while for us to notice some simple cases which frequently occur in practice and which are included under it. We mention here three such cases each of which is included as a special case under the next following one:

(a) Condition (A) is fulfilled if p, q, r are continuous functions of (x,λ) throughout (I, L).

(b) Condition (A) is fulfilled if p, q, r are continuous functions of (x, λ) throughout (I, L) except when $x = c_1, c_2, \cdots c_k$; and a positive constant M exists such that except at the last mentioned points $|p| < M$, $|q| < M$, $|r| < M$.

(c) Condition (A) is fulfilled if except when $x = c_1, c_2, \cdots c_k$

$$p = f(x)\ \phi_1(x,\lambda), \qquad q = f(x)\ \phi_2(x,\lambda), \qquad r = f(x)\ \phi_3(x, \lambda),$$

where ϕ_1, ϕ_2, ϕ_3 are continuous functions of (x,λ) throughout (I, L); and

$$f(x) = (x - c_1)^{a_1}(x - c_2)^{a_2} \cdots (x - c_k)^{a_k},$$

a, $\cdots$ a_k being real constants greater than -1.

4. Equations whose coefficients are polynomials in λ. By imposing further restrictions on p, q, r, γ, γ_1 we could make the functions y and y' have certain further properties, for instance that of possessing first partial derivatives with regard to λ which are continuous functions of (x, λ). Without stopping to treat these intermediate possibilities, we will consider what in a sense is an extreme case, and yet one sufficiently general to cover the cases which most frequently occur in practice. The theorem is this:

If c is any point in (I), *and* γ, γ_1 *any constants, and if* p, q, r *are polynomials in* λ *whose coefficients are integrable (and in the case of* p *and* q *absolutely integrable) functions of* x *throughout* (I), *then,* $y(x, \lambda)$ *being the solution of* (15), *which satisfies the auxiliary conditions*

$$y(c, \lambda) = \gamma, \qquad y'(c, \lambda) = \gamma_1,$$

$y(x, \lambda)$ *and* $y'(x, \lambda)$ *are continuous functions of* (x, λ) *when* x *lies in* (I) *and* λ *has any value real or complex. Furthermore,* x_0 *being any point of* (I), $y(x_0, \lambda)$ *and* $y'(x_0, \lambda)$ *are integral analytic functions of* λ.*

In order to prove this let us consider a region (Λ) in the complex λ-plane consisting of all the points within and on the boundary of an arbitrarily chosen circle. The functions p, q, r obviously satisfy condition (A) of §3 if in this condition the region (Λ) is substituted for the interval (L). Accordingly we see, from the proof there given, that $y(x, \lambda)$ and $y'(x, \lambda)$ are continuous functions of (x, λ) throughout (I, Λ). Moreover, the series (4) and (5) being uniformly convergent in (I, Λ), will, when we let $x = x_0$, be uniformly convergent in (Λ). From formulæ (3) and (6) we see that the terms of (4) and (5) are polynomials in λ. We see therefore by a fundamental theorem of Weierstrass* that $y(x_0, \lambda)$ and $y'(x_0, \lambda)$ are in (Λ) single valued analytic functions with no singular points. From these facts our theorem follows since the circle (Λ) may be taken as large as we please.

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* That is, analytic functions having no finite singular points.

Finally we note that the assumption made in the last theorem that γ and γ_1 are constants is not essential. For if we denote by y_1 and y_2 the solutions of (15) which satisfy the auxiliary conditions:

$$\begin{aligned} y_1(c, \lambda) &= 1, & y_2(c, \lambda) &= 0, \\ y_1'(c, \lambda) &= 0, & y_2'(c, \lambda) &= 1, \end{aligned}$$

the solution y which satisfies the conditions

$$y(c, \lambda) = \gamma(\lambda), \qquad y'(c, \lambda) = \gamma_1(\lambda)$$

may be written

$$y = \gamma y_1 + \gamma_1 y_2.$$

If then γ and γ_1 are single valued analytic functions with no singular points in a certain continuum (Λ_0) of the λ-plane, we see at once that $y(x, \lambda)$ and $y'(x, \lambda)$ are continuous functions of (x, λ) throughout (I, Λ_0), since by the last theorem this is true of y_1, y_1', y_2, y_2'. For the same reason if x_0 is any point in (I), $y(x_0, \lambda)$ and $y'(x_0, \lambda)$ are single valued analytic functions without singular points in (Λ_0).

5. **Roots of functions which involve parameters.** Let us consider a real function $\phi(x,\lambda)$ defined for all values of x in (I) when λ belongs to a certain continuous or discontinuous range of values. We suppose that ϕ satisfies the following conditions:

(*a*) ϕ has a first derivative with regard to x, $\phi'(x, \lambda)$, which is continuous throughout (I) for all values of λ with which we are concerned.

(*b*) As λ approaches a certain value l (either continuously or in some

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*If in a connected two dimensional region of the complex plane a series converges uniformly, and if throughout this region its terms are single valued analytic functions without singular points, then the same will be true of the function represented by the series. *Cf.* for instance Burkhardt, *Analytische Functionen*, 1st edition p. 139.

discontinuous manner as the case may be*), $\phi(x, \lambda)$ and $\phi'(x, \lambda)$ approach uniformly certain limits $f(x)$ and $f'(x)$.

(*c*) The functions $f(x)$ and $f'(x)$ do not both vanish at any point of (I).

(*d*) $f(x)$ vanishes neither at a nor at b.

It should be noticed that it follows from (*a*) and (*b*) that $f(x)$ and $f'(x)$ are continuous throughout (I), and that f' is the derivative of f. From (*c*) it follows that $f(x)$ can not vanish at an infinite number of points in (I), for otherwise these points would have at least one limiting point in (I) at which both f and f' would vanish.

THEOREM. *If $\phi(x, \lambda)$ satisfies conditions (a), (b), (c), (d) and if its limit $f(x)$ has just k roots $x_1, \cdots x_k$ in (I), then for all values of λ sufficiently near to l, $\phi(x, \lambda)$ has just k roots in (I), and these roots approach the values $x_1, \cdots x_k$ as λ approaches l.*

In order to prove this let us suppose the roots $x_1, \cdots x_k$ to be arranged in order of magnitude $(a < x_1 < x_2 < \cdots < x_k < b)$, and consider the k intervals

$$(I_i) \qquad |x - x_i| \leqq \delta \qquad (i = 1, 2, \cdots k),$$

where δ is a positive real constant which we suppose taken so small that all the intervals (I_i) lie in (I), and that no two of them have a point in common. Since $f'(x)$ is continuous and, by (*c*), does not vanish at any of the points x_i, δ may be taken so small that $f'(x)$ does not vanish in any of the intervals (I_i). This we suppose done.

We define further the intervals J as follows:

$$(J_0) \qquad a \leqq x \leqq x_1 - \delta,$$

$$(J_i) \qquad x_i + \delta \leqq x \leqq x_{i+1} - \delta \qquad (i = 1, 2, \cdots k - 1),$$

$$(J_k) \qquad x_k + \delta \leqq x \leqq b.$$

Our theorem will be proved if we can show that no matter how small δ may be chosen, a positive ϵ can be taken so small that for every value of λ (in the range with which we are concerned) for which $|\lambda - l| < \epsilon$†, the function $\phi(x,\lambda)$ has no roots in any of the intervals (J_i) and just one root in each of the intervals (I_i). To prove this let us denote by m_1 the smallest value which $|f'(x)|$ has in any of the intervals (I_i), and take ϵ so small that when $|\lambda - l| < \epsilon$

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*For instance we might have $l = +\infty$ and the range of values of λ with which we are concerned might be the positive integers.

†If $l = \infty$ we must replace this inequality by $|1/\lambda| < \epsilon$.

the inequality $| \phi'(x, \lambda) - f'(x) | < m_1$ is satisfied. This makes it certain that $\phi'(x, \lambda)$ does not vanish in any of the intervals (I_i) when $|\lambda - l| < \epsilon$. Now denote by m_2 the smallest value of $|f(x)|$ in the intervals (J_i), and take ϵ still smaller than before, if necessary, so that when $|\lambda - l| < \epsilon$ the inequality $|\phi(x,\lambda) - f(x)| < m_2$ is fulfilled; thus making it certain that, for these values of $\lambda, \phi(x, \lambda)$ does not vanish in any of the intervals (J_i), and has the same sign throughout each of them that $f(x)$ has there. But $f(x)$ changes sign at every root x_i, since $f'(x_i) \neq 0$; accordingly, when $|\lambda - l| < \epsilon$, the sign of $\phi(x, \lambda)$ is alternately plus and minus in the intervals $(J_0), (J_1), \cdots$. Thus we see that $\phi(x, \lambda)$, which has no roots in the intervals (J_i), has at least one root in each of the intervening intervals (I_i). It cannot have more than one root in any interval (I_i), for otherwise $f'(x)$ would vanish there by Rolle's theorem.

We will now apply the theorem which has just been proved to the solutions of linear differential equations of the second order, confining ourselves, however, to the case of homogeneous equations, since here the mere requirement that the solution is not identically zero is sufficient to ensure its satisfying condition (c).

If in the equation

$$\frac{d^2y}{dx^2} + p(x, \lambda) \frac{dy}{dx} + q(x, \lambda) y = 0$$

p and q satisfy conditions (A) of §3, and if $y(x, \lambda)$ is the solution of this equation which satisfies the conditions

$$y(a, \lambda) = \gamma(\lambda), \qquad y'(a, \lambda) = \gamma_1(\lambda),$$

where γ and γ_1 are continuous throughout (L), and if, for all values of λ in (L), $y(x, \lambda)$ has at least k roots in the interval $a < x \leqq b$ which arranged in order of magnitude we denote by $x_1(\lambda), x_2(\lambda), \cdots$, then $x_1, \cdots x_k$ are throughout (L) continuous functions of λ provided one or the other of the following conditions is fulfilled:

(1) $\gamma(\lambda)$ *does not vanish in* (L);

(2) $\gamma(\lambda) \equiv 0$, *and* $\gamma_1(\lambda)$ *does not vanish in* (L).*

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*The object of these last two conditions is to prevent $x_1(\lambda)$ from reaching the value a This may be attained more generally by the following condition:

(3) γ and γ_1 do not both vanish at any point of (L), and the fraction γ_1/γ has, for values of λ for which $\gamma \neq 0$, if any such exist, a finite lower limit, possibly negative.

The cases mentioned in the text are, however, by far the most important.

We prove this theorem first in case (1). Here we have merely to prove that if l is any value of λ in (L) the functions $x_1, \cdots x_k$ are continuous when $\lambda = l$. Suppose that $y(x, l)$ has just $k + m$ roots in (I). If $y(b, l) \neq 0$ our theorem follows from the one last proved which tells us that when λ is sufficiently near to l, $y(x,\lambda)$ also has exactly $k + m$ roots in (I), and that as λ approaches l, $x_1(\lambda), \cdots x_{k+m}(\lambda)$ approach respectively $x_1(l), \cdots x_{k+m}(l)$. If, however, $y(b, l) = 0$ we must extend the interval (I) a little beyond the point b by defining p and q for values of x a little greater than b. For instance we may let $p(x, \lambda) = q(x,\lambda) = 0$ when $x > b$. If $b' > b$, $y(b', l) \neq 0$ since $y(b, l) = 0$. Accordingly we may apply the first theorem of this section to the interval $a \leqq x \leqq b'$, and we see that, for values of λ sufficiently near to l, $y(x, \lambda)$ has just $k + m$ roots in this interval, and that these roots approach $x_1(l), \cdots x_{k+m}(l)$ as their limits as λ approaches l. Since k of these x's always lie in (I) by hypothesis our theorem is proved.

It remains to prove the theorem when $\gamma(\lambda) \equiv 0$. Here we determine a constant $a' > a$ but differing from a so little that $y(x, \lambda)$ does not vanish in the interval $a < x \leqq a'$ for any value of λ in (L). This we do by taking a' in such a way that $y'(x, \lambda)$ does not vanish in the interval $a \leqq x \leqq a'$, for then y which vanishes when $x = a$ could not vanish again by Rolle's theorem. By applying to the interval $a' \leqq x \leqq b$ the same reasoning which was before applied to (I), the truth of our theorem follows.

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